

**Asymptotic Freedom and Infrared slavery in PT-symmetric
Quantum Electrodynamics (*)**

R.Acharya (*)

Physics Department, Arizona State University, Tempe, AZ 85287

and

P. Narayana Swamy (†)

Physics Department, Southern Illinois University, Edwardsville, IL 62026

Abstract

We establish that there is no finite PT-symmetric Quantum Electrodynamics (QED) and as a consequence the Callan-Symanzik function $\beta(\alpha) < 0$, for all α greater than zero: PT-symmetric QED exhibits both asymptotic freedom and infrared slavery.

PACS numbers: 11.30.Er 12.20.-m 11.10.Lm

(†): Professor Emeritus, e-mail address: pswamy@siue.edu

(*): Professor Emeritus, e-mail address: Raghunath.acharya@asu.edu

June 2005

(*) To the memory of Ken Johnson: his masterful presentations at Harvard, Bangalore and Dublin had been a rewarding experience for both of us.

1. Introduction

In an earlier publication, hereafter referred to as I [1], we have demonstrated the absence of a finite eigenvalue in QED with massless electron, thereby confirming the result established by Johnson, Baker, Callan et al [2-6], in the 1970's. The conclusion arrived at in I was based on a non-perturbative “gauge technique” pioneered by Salam [7] and by Ball and Zachariasen [8].

In this note we shall simply transcribe the conclusions of I, to arrive at an identical conclusion in the PT-symmetric QED, formulated by Bender and co-workers [9]. Bender's PT-QED is obtained by replacing the sign of the fine structure constant, $\alpha = e^2/4\pi$ by $-\alpha$. This theory remains PT-symmetric and, more importantly, is asymptotically free (AF) ! Our concern in this note is to address the all important question, namely, is PT-QED also a confining theory? As a first step in tackling this difficult problem, we shall establish that PT-QED exhibits infrared (IR) slavery, i.e., the running coupling $\alpha(q^2)$ increases without bound in the infrared limit, $q^2 \rightarrow 0$. It is now standard lore i.e., the conjecture of Weinberg [10], Fritzsche, Gell-Mann and Leutwyler and Gross-Wilczek, then asserts that “infrared slavery is responsible for confinement”. In the words of Weinberg: *“the decrease of the coupling constant at high energy or short distance of course implies an increase at low energy or long distance”*.

Our analysis which closely follows our earlier work in I and the conclusion stated above relies on establishing the absence of a finite eigenvalue of α i.e., the PT-QED which is formally obtained by switching the sign of α does not possess a non-trivial IR fixed point and consequently does not exhibit a coulomb phase ! In other words, the Callan-Symanzik function of PT-QED, $\beta(\alpha)$ as a function of α remains negative for all values of α , for $\alpha > 0$ and never turns over. This conclusion then reinforces the deep observation of Witten et al [11], “in a conformal field theory, any interacting field strength must couple both to electrons and to monopoles. In particular, QED without elementary monopoles cannot have a non-trivial fixed point”. Witten's remark applies equally well to PT-QED.

Since our entire analysis of PT-QED parallels our earlier exhaustive discussion of QED in I, line by line, we shall highlight the salient steps of I (in order to keep the discussion self-contained), with the important clarification of the role of the QED transverse vector vertex: in I, we had not explicitly spelled out the precise forms of the transverse vertex; we do so here. Throughout our discussion, we will set the “photon” propagator to its free form. This is justified since, here, $\beta(\alpha) = \gamma_A(\alpha)$, where γ_A is the anomalous dimension of the “photon” and therefore $\beta(\alpha) = 0$ at the fixed point, hence the “photon” propagator is bare. The other important point is the observation (see Bernstein, [12]) that PT invariance will suffice to yield the standard form of the electron propagator, as discussed in Sec.2 below.

2. Dyson-Schwinger equation at QED fixed point

We begin by reviewing the salient points of our earlier work, I. Let us start with the standard finite QED and introduce the PT symmetric extension by replacing $\alpha = e^2/4\pi$ by $-\alpha$. Indeed the correct starting point for a PT symmetric QED is the form of the renormalized electron propagator whose general invariant form

$$S^{-1}(p) = \not{p} A(p^2) + \Sigma(p^2), \quad (1)$$

follows from Lorentz invariance and parity invariance. As a matter of fact PT invariance will suffice to establish the above invariant form, as has been emphasized by Bernstein [12]. We shall review the theory of finite QED in order to provide the appropriate notations even as it necessitates repeating some material from ref.(1). In the finite QED theory, we set $\Sigma(p^2) \equiv 0$, for any p and chiral symmetry remains an exact symmetry. The proper renormalized vertex function in QED satisfies the Ward-Takahashi identity

$$(p - p')^\mu \Gamma_\mu(p, p') = S^{-1}(p) - S^{-1}(p'). \quad (2)$$

We would like to refer the reader to our earlier work [13] where we have reviewed the well-known consequences of the Gell-Mann-Low eigenvalue equation $\psi(x) = 0$, which may or may not have a non-trivial zero at $x_0 = \alpha_0$ at the position of the bare fine structure constant of QED, $\alpha_0 = Z_3^{-1}\alpha$. The important premise of the finite theory of QED is that the position of the zero, $x = x_0$ can be determined by working with QED with zero physical mass [2]. This is predicated upon the application of Weinberg's theorem [14], which ensures that terms vanishing asymptotically in each order of perturbation theory in the massive case do not sum to dominate over the asymptotic parts.

It can be easily shown that at the Gell-Mann-Low fixed point with $m = 0$ in finite QED, the full, exact, renormalized electron propagator has the simple scaling form [15]

$$S^{-1}(p) = \not{p} A(p^2) = \not{p} \left(\frac{p^2}{\mu^2} \right)^\gamma \quad (3)$$

where $\gamma(\alpha)$ is the anomalous dimension of the electron in the massless theory given in the Landau gauge by

$$\gamma = \mu \frac{\partial}{\partial \mu} \ln Z_2 = O(\alpha^2) + \dots \quad (4)$$

and μ is the subtraction point. This can be established as follows. Starting with the Callan-Symanzik renormalization group equation [16], if we specialize to the Landau gauge, set $m = 0$

(massless electron) and $\beta(\alpha) = 0$ at the fixed point, then we have the equation satisfied by the two-point function, essentially the inverse electron propagator

$$\left(\mu \frac{\partial}{\partial \mu} + 2\gamma\right) \Gamma^{(2)}(p, \alpha, 0, \mu, 0) = 0. \quad (5)$$

The solution for the two-point function can be expressed as

$$A(p^2) = \left(\frac{p^2}{\mu^2}\right)^\gamma. \quad (6)$$

which is customarily expressed in terms of Euclidean momenta.

This can be confirmed by examining the trace anomaly in QED [17]. At a fixed point, $\beta(\alpha) = 0$, when we set the physical electron mass equal to zero, the divergence of the scale current is given by

$$\partial^\mu D_\mu = \frac{\beta(\alpha)}{2\alpha} F_{\mu\nu} F^{\mu\nu} + [1 + \gamma_\theta(\alpha)] m \bar{\psi} \psi = 0 \quad (7)$$

and hence scale invariance is exact. Assuming that scale invariance is not spontaneously broken, $Q_D|0\rangle = 0$, where

$$Q_D = \int d^3x D_0(\mathbf{x}, t), \quad (8)$$

(*i.e.*, no dilatons are present in QED) from which Eq.(3) follows.

Let us now consider the Dyson-Schwinger equation satisfied by the inverse of the full, exact, renormalized electron propagator in massless, PT symmetric QED:

$$S^{-1}(p) = Z_2 \not{p} + i Z_2 e^2 (2\pi)^{-4} \int d^4k \gamma^\mu D_{\mu\nu} S(k) \Gamma^\nu(p, k). \quad (9)$$

It is well-known that at the fixed point $\beta(\alpha) = 0$, in a theory of $m = 0$, spin- $\frac{1}{2}$ QED, the full, exact, renormalized “photon” propagator is given exactly by the free “photon” propagator, as established by Eguchi [18] and thus we have in the Landau gauge:

$$D_{\mu\nu}(q) = \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) \frac{1}{q^2}, \quad (10)$$

where $q_\mu = p_\mu - k_\mu$. This is so since $\beta(\alpha) = \gamma_A(\alpha) = 0$, where $\gamma_A(\alpha) = \mu(\partial/\partial\mu) \ln Z_3$ is the anomalous dimension of the “photon”. The solution to the Ward-Takahashi identity satisfied by

the renormalized, proper vector vertex function in QED can be determined in the standard manner by the gauge technique [7] to yield the form

$$\Gamma^\nu = \Gamma_L^\nu + \Gamma_T^\nu, \quad (11)$$

where the longitudinal part of the vertex function admits the general, kinematical singularity-free solution [8, 19] given by

$$\begin{aligned} \Gamma_L^\nu(p, k) = & \frac{1}{2}(A + \tilde{A})\gamma^\nu + \frac{\Sigma - \tilde{\Sigma}}{k^2 - p^2}(\gamma^\nu \not{p} + \not{k}\gamma^\nu) \\ & - \frac{1}{2}\left(\frac{A - \tilde{A}}{k^2 - p^2}\right) [2\not{p}\gamma^\nu \not{k} + (p^2 + k^2)\gamma^\nu], \end{aligned} \quad (12)$$

and the transverse piece obeys the condition

$$(p - p')_\mu \Gamma_T^\mu(p, p') = 0, \quad (13)$$

and consequently undetermined by the Ward-Takahashi identity. Here we have employed the notation $\tilde{A} = A(k^2)$ etc. It must be stressed that we shall retain the transverse piece throughout our calculation. Indeed, undetermined and arbitrary as it is, the transverse piece of the vertex has a significant bearing on our conclusions. In massless finite QED, when the chiral symmetry is exact, the above solution for the vertex function reduces to

$$\Gamma^\nu(p, k) = \frac{1}{2}(A + \tilde{A})\gamma^\nu - \frac{1}{2}\left(\frac{A - \tilde{A}}{k^2 - p^2}\right) [2\not{p}\gamma^\nu \not{k} + (p^2 + k^2)\gamma^\nu] + \Gamma_T^\nu(p, k), \quad (14)$$

and the Dyson-Schwinger equation, Eq.(9) reduces to

$$S^{-1}(p) = \not{p}A(p^2) = Z_2\not{p} + iZ_2e^2(2\pi)^{-4} \int d^4k \left(\frac{\not{p}q_\nu}{q^2} - \gamma_\nu \right) \frac{1}{q^2} \frac{\not{k}}{k^2\tilde{A}} [\Gamma_L^\nu(p, k) + \Gamma_T^\nu(p, k)]. \quad (15)$$

We shall now address the transverse part of the vertex function. The general invariant form of the transverse part has been determined by Ball and Chiu [8]. In the special case of massless electron, only four of the basic functions contribute [20] and accordingly, we have

$$\Gamma_T^\nu(p, k, q) = \sum_{i=2,3,6,8} \tau_i(p^2, k^2, q^2) T_i^\nu(p, k), \quad (16)$$

where the $\tau_i(p^2, k^2, q^2)$ are unknown, arbitrary functions and the invariant functions are listed below:

$$\begin{aligned} T_2^\nu &= (p^\nu k \cdot q - k^\nu p \cdot q) (\not{k} + \not{p}), \\ T_3^\nu &= q^2 \gamma^\nu - q^\nu \not{q}, \\ T_6^\nu &= \gamma^\nu (k^2 - p^2) - (k + p)^\nu (\not{k} - \not{p}), \\ T_8^\nu &= \gamma^\nu \sigma_{\lambda\nu} k^\lambda p^\nu - k^\nu \not{p} + p^\nu \not{k}. \end{aligned} \quad (17)$$

We may now evaluate the trace over the Dirac matrices after multiplying by \not{p} and dividing by $4p^2$. After a tedious computation, we obtain the following result:

$$\begin{aligned} A(p^2) - Z_2 &= -iZ_2 e^2 (2\pi)^{-4} \int d^4k \frac{1}{p^2 k^2 A(k^2)} \\ &\left\{ \frac{1}{2q^4} (A + \tilde{A}) [2p^2 k^2 - (p \cdot k)(p^2 + k^2)] + \frac{1}{q^4} \frac{A - \tilde{A}}{k^2 - p^2} (p \cdot k)(p^2 - k^2)^2 \right. \\ &\left. - (A + \tilde{A}) \frac{(p \cdot k)}{q^2} - \frac{(A - \tilde{A})}{q^2(k^2 - p^2)} [4(p \cdot k)^2 - (p \cdot k)(p^2 + k^2)] + \Gamma_T^1(p^2, k^2, p \cdot k) \right\}, \end{aligned} \quad (18)$$

where Γ_T^1 arises from the trace calculation of the term containing the transverse vertex piece and is given by the expression

$$\begin{aligned} &\frac{1}{q^4} \{ \tau_2 \quad p \cdot k(p^2 - k^2) [k^2(p^2 - p \cdot k) + 2p \cdot k(p^2 - p \cdot k) - p^2(p \cdot k - k^2)] \\ &\quad + \tau_6(p^2 - k^2) [k^2(p^2 - p \cdot k) - 2p \cdot k(p^2 - p \cdot k) + p^2(p \cdot k - k^2)] \\ &\quad - \tau_8 \frac{1}{2} (k^2 - p \cdot k) [2p \cdot k(p^2 - p \cdot k) - p^2(p \cdot k - k^2)] - \tau_8 \frac{1}{2} k^2(p^2 - p \cdot k)^2 \} \\ &+ \frac{1}{q^2} \{ 2\tau_2 p^2 k^2 (p \cdot k - k^2)(p \cdot k) [p^2(p \cdot k - k^2) - k^2(p^2 - p \cdot k)] + 2\tau_2 (p \cdot k)^2 (p \cdot k - k^2) \\ &\quad + 3\tau_3 (p \cdot k) \\ &\quad + \tau_6 [p \cdot k(k^2 - p^2) + 2\{(p \cdot k)^2 - p^2 k^2\} + 3(p \cdot k)^2 k^2 - p^2] \\ &\quad + 3\tau_8 [(p \cdot k)^2 - p^2 k^2] \}. \end{aligned} \quad (19)$$

$$(20)$$

This non-linear integral equation satisfied by the function $A(p^2)$ is the exact consequence of the Dyson-Schwinger equation for the electron propagator in finite QED at the Gell-Mann-Low fixed point since we have not introduced any approximations and we have not discarded the transverse piece. Transforming to Euclidean momenta by implementing the transformations:

$$d^4k \rightarrow id^4k = i \int k^3 dk d\Omega; \quad p^2 \rightarrow -p^2; \quad k^2 \rightarrow -k^2; \quad p \cdot k \rightarrow -p \cdot k. \quad (21)$$

and after performing the angular integrals, we obtain

$$\begin{aligned}
A(-p^2) - Z_2 &= Z_2 e^2 (2\pi)^{-4} \int k^3 dk \frac{1}{p^2 k^2 A(-k^2)} \\
&\quad \left\{ \frac{1}{2} (A + \tilde{A}) \left[2p^2 k^2 I_4 - (p^2 + k^2) I_5 \right] - (p^2 - k^2) (A - \tilde{A}) I_5 \right. \\
&\quad \left. - (A + \tilde{A}) I_2 - \frac{(A - \tilde{A})}{(p^2 - k^2)} \left[4I_3 - (p^2 + k^2) I_2 \right] + \tilde{\Gamma}_T^1(-p^2, -k^2) \right\}, \quad (22)
\end{aligned}$$

where

$$\tilde{\Gamma}_T^1(-p^2, -k^2) = \int d\Omega \Gamma_T^1(-p^2, -k^2, p \cdot k). \quad (23)$$

We should recall that due to the dependence on $p \cdot k$ and the fact that the invariant functions τ_i are arbitrary and unknown, the angular integration of the transverse part cannot be explicitly carried out in the *exact* theory. The term containing the transverse part, namely $\tilde{\Gamma}_T^1(-p^2, -k^2)$ is therefore an undetermined quantity. We have left it in this form only to indicate that it contains nonzero terms and henceforward we shall denote the consequences of this transverse vertex part appropriately as $\tilde{\Gamma}_T^2, \tilde{\Gamma}_T^3$ etc. It must be stressed that the transverse parts *must* survive: their presence is essential for multiplicative renormalizability of the propagator [21]. The quantities $I_1, I_2 \dots$ are angular integrals listed in our earlier work, [1]. All momenta are Euclidean, defined by $p_E = \sqrt{-p^2}$, $k_E = \sqrt{-k^2}$ and in what follows we shall drop the subscript E for Euclidean momenta in order to avoid clutter. Making use of the results in Appendix A of our earlier work [1,22], we arrive at the following result

$$\begin{aligned}
A(p^2) - Z_2 &= -\frac{Z_2 e^2}{16\pi^2} \int_0^\infty k dk \frac{1}{p^2 A(k^2)} \\
&\quad \left\{ \frac{A(p^2) - A(k^2)}{(p^2 - k^2)} \left[\frac{2\sigma^2(p^2 - k^2)^2}{p_{>}^2(1 - \sigma^2)} - p_{<}^2(1 + \sigma^2) \right] + \tilde{\Gamma}_T^2(p^2, k^2) \right\}, \quad (24)
\end{aligned}$$

where $\tilde{\Gamma}_T^2(p^2, k^2)$ arises from the transverse vertex part, $\sigma = p_{<}/p_{>}$ and $p_{<} = \min\{p, k\}$, $p_{>} = \max\{p, k\}$. This result is a consequence of the essential ingredients of finite QED, with no approximations nor additional assumptions. In order to ascertain whether this finite theory of QED, as we have developed thus far, admits of a solution to the Gell-Mann-Low eigenvalue equation, we proceed as follows. The self-consistency of the theory constructed in this manner, of massless QED at the fixed point can be checked by making the replacement

$$A(p^2) = \left(\frac{p^2}{\mu^2} \right)^\gamma \quad (25)$$

in accordance with Eq.(3) where $\gamma = \gamma(\alpha)$ is the QED anomalous dimension in the $m = 0$ theory. After some algebra, we thus obtain the result

$$\begin{aligned} \left(\frac{p^2}{\mu^2}\right)^\gamma - Z_2 &= -\frac{Z_2 e^2}{16\pi^2} \int_0^p k dk \left\{ [(p^2/k^2)^\gamma - 1] \frac{(p^2 k^2 - 3k^4)}{p^4(p^2 - k^2)} + \tilde{\Gamma}_T^3(p^2, k^2) \right\} \\ &\quad - \frac{Z_2 e^2}{16\pi^2} \int_p^\infty k dk \left\{ [(p^2/k^2)^\gamma - 1] \frac{(p^2 k^2 - 3p^4)}{p^2 k^2(p^2 - k^2)} + \tilde{\Gamma}_T^4(p^2, k^2) \right\}, \end{aligned} \quad (26)$$

where $\tilde{\Gamma}_T^3$ and $\tilde{\Gamma}_T^4$ represent the contributions arising from the transverse piece vertex function. With a change in variables, $s = p^2, k^2 = sx$, this can be rewritten in the form

$$\begin{aligned} (s/\mu^2)^\gamma - Z_2 &= -\frac{Z_2 e^2}{32\pi^2} \int_0^1 \left\{ dx \frac{x(1-3x)(x^{-\gamma} - 1)}{(1-x)} + \Gamma_T^5(s, x) \right\} \\ &\quad - \frac{Z_2 e^2}{32\pi^2} \int_1^\infty \left\{ dx \frac{x(x-3)(x^{-\gamma} - 1)}{(1-x)} + \Gamma_T^6(s, x) \right\}, \end{aligned} \quad (27)$$

where Γ_T^5 and Γ_T^6 (suppressing the tilde henceforward) are contributions arising from the transverse vertex piece. For general values of γ , we can evaluate the integrals [1] and we obtain the result

$$\begin{aligned} (s/\mu^2)^\gamma - Z_2 &= -\frac{Z_2 e^2}{32\pi^2} \{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\ &\quad - 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\ &\quad + F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_T^7(s) \} \end{aligned} \quad (28)$$

in terms of the hypergeometric functions, where $\Gamma_T^7(s)$ arises from the integral of the contribution from the transverse vertex piece.

If we evaluate Eq.(28) at $s = \mu^2$, we obtain

$$\begin{aligned} Z_2 = 1 &+ \frac{Z_2 e^2}{32\pi^2} \{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\ &\quad - 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\ &\quad + F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_T^7(\mu^2) \}, \end{aligned} \quad (29)$$

which can be rewritten as

$$\begin{aligned} Z_2^{-1} = 1 &- \frac{e^2}{32\pi^2} \{ 3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\ &\quad - 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\ &\quad + F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_T^7(\mu^2) \}, \end{aligned} \quad (30)$$

This has been obtained in the Landau gauge to all orders in α . We are now ready to analyze the results contained in Eqs.(28, 30), a major consequence of the Dyson-Schwinger equations of finite QED with massless electron at a fixed point, specifically in the PT symmetric QED. Care is required in handling the hypergeometric functions appearing in these equations and the reader is referred to Appendix B of our earlier work [1].

3. Conclusion and Summary

Let us examine Eq.(28) and determine what are the allowed values of γ , the anomalous dimension in massless QED. From Eqs.(28) and (29), we obtain

$$\begin{aligned}
\left(s/\mu^2\right)^\gamma &= 1 + \frac{Z_2 e^2}{32\pi^2} \{3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\
&- 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\
&+ F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_T^7(\mu^2)\} \\
&- \frac{Z_2 e^2}{32\pi^2} \{3F(1, 3, 4; 1) - F(1, 2, 3; 1) + F(1, 2 - \gamma, 3 - \gamma; 1) \\
&- 3F(1, 3 - \gamma, 4 - \gamma; 1) + 3F(1, -1, 0; 1) - F(1, -2, -1; 1) \\
&+ F(1, \gamma - 2, \gamma - 1; 1) - 3F(1, \gamma - 1, \gamma; 1) + \Gamma_T^7(s)\}, \tag{31}
\end{aligned}$$

which simplifies to

$$\left(s/\mu^2\right)^\gamma = 1 + \frac{Z_2 \alpha}{8\pi} \left\{ \Gamma_T^7(\mu^2) - \Gamma_T^7(s) \right\}. \tag{32}$$

We recall that the necessary and sufficient condition for finite Z_2 and finite Z_2^{-1} is $\gamma = 0$. That is, if $\gamma = 0$, then both Z_2 and Z_2^{-1} are finite. Conversely, if Z_2 and Z_2^{-1} are both finite, then γ must vanish. This assertion follows from the defining relation, Eq.(4). In such a finite field theory, all the three renormalization constants, m_0 , Z_2 , and Z_3 tend to finite limits in the limit of infinite cut-off parameter Λ . We can then observe that our stated result in Landau gauge, with $\gamma = 0$, yields

$$Z_2^{-1}(\mu^2, \xi = 0) = 1 - \frac{\alpha}{8\pi} \Gamma_T^7(\mu^2). \tag{33}$$

The divergences present in the longitudinal pieces clearly cancel only if $\gamma = 0$, for an *arbitrary* choice of the transverse piece $\Gamma_T^7(\mu^2)$. This has been demonstrated in Appendix B of our earlier work [1].

From Eq. (33), we then conclude that $\Gamma_T^7(\mu^2)$ must also be finite when $\gamma = 0$. Furthermore, Eq.(32) simplifies to

$$Z_2(\xi = 0) \frac{\alpha}{8\pi} \left\{ \Gamma_T^7(\mu^2) - \Gamma_T^7(s) \right\} = 0. \quad (34)$$

From Eqs.(33) and (34) we finally conclude that

$$e^2 \equiv 0, \quad (35)$$

since $Z_2(\xi = 0)$ cannot vanish (Z_2^{-1} must be finite) and the transverse piece vertex *cannot* be a constant independent of s . In other words, the only unphysical manner that one can arrive at a non-trivial eigenvalue $e^2 \neq 0$ is if the transverse vertex piece is a constant, but it is not.

We can provide an alternative demonstration of the trivial eigenvalue result directly from Eq.(33) as follows. From the definition of $\gamma(\alpha)$, we thus obtain

$$\gamma(\alpha) = -\mu \frac{\partial}{\partial \mu} \ln \left(1 - \frac{\alpha}{8\pi} \Gamma_T^7(\mu^2) \right) = \frac{1}{8\pi(1 - \frac{\alpha}{8\pi} \Gamma_T^7(\mu^2))} \left\{ \beta(\alpha) \Gamma_T^7(\mu^2) + \alpha \mu \frac{\partial \Gamma_T^7(\mu^2)}{\partial \mu} \right\}. \quad (36)$$

From Eq.(33), Z_2^{-1} is finite, $\beta(\alpha) = \mu(\partial\alpha/\partial\mu) = 0$ at the fixed point, and it thus follows that

$$\gamma(\alpha) = \frac{\alpha}{8\pi} Z_2 \mu \frac{\partial \Gamma_T^7(\mu^2)}{\partial \mu}. \quad (37)$$

Hence $\gamma(\alpha) = 0$ implies $\alpha \equiv 0$, μ arbitrary and $\partial \Gamma_T^7(\mu^2)/\partial \mu \neq 0$.

We have carried out our investigation in the Landau gauge. The form of the solution $A(p^2) = (p^2/\mu^2)^{\gamma(\xi)}$ must remain valid in all covariant gauges, $\xi \neq 0$, in the minimal subtraction scheme [23] at $\beta(\alpha) = 0$. It remains to investigate the non-perturbative gauge technique in a general setting and determine the choice of ξ which will accomplish the task of establishing a finite field theory non-perturbatively, i.e., to all orders in α . This task will occupy us in a forthcoming work.

It may be important to emphasize that what we have examined is non-perturbative QED and our conclusions are based on non-perturbative techniques developed for investigating the Dyson-Schwinger equation in QED, now extended to include the PT symmetric field theory.

The PT symmetric QED is a theory with asymptotic freedom. We have demonstrated that the trivial eigenvalue $\alpha = 0$ is the only solution. The beta function starts out negative, $\beta(\alpha) < 0$, at the origin, remains negative for all $\alpha > 0$ and does not turn over to positive values since the beta

function cannot have an infrared fixed point. We therefore conclude that the PT symmetric QED is a theory which incorporates both asymptotic freedom and infrared slavery.

The central result that $\beta(\alpha) < 0$ for all $\alpha > 0$ may be contrasted with the circumstance in non-supersymmetric Quantum Chromodynamics (QCD) with a large number of flavors N_F , where a non-trivial fixed point of $\beta(\alpha)$ does exist and the theory is in a non-Abelian coulomb phase [24]. in contrast, here in PT-QED, it is expected that infrared slavery will lead to the generation of dynamical “photon” mass, i.e., the theory will exhibit a *mass gap*. We shall address this issue in a separate publication.

References and Footnotes

1. R. Acharya and P. Narayana Swamy, Int. J. Mod. Phys.**A** **12** 3799 (1997).
2. K. Johnson and M. Baker, Phys. Rev. **D8** 1110 (1973) and references cited therein; M. Baker and K.Johnson, Physica **A** **96**, 120 (1979); S.L.Adler, C.Callan, R.Jackiw and D.Gross, Phys.Rev. **D6**, 2982 (1972). See also N. Krasnikov, Phys.Lett. **B225**, 284 (1989).
3. S.L.Adler, Phys. Rev. **D5**, 3021 (1972); J. Bernstein, Nucl.Phys.**B95**, 461 (1975). See also S.L.Adler, preprint arXiv:hep-ph/0505177, May 2005.
4. M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954); N.N.Bogoliubov and D.Shirkov, *Introduction to the theory of quantized fields*, Interscience, New York, 1959; K. Wilson, Phys. Rev. **D3**, 1818 (1971).
5. J.S.Bell and R.Jackiw, Nuovo Cimento **A60**, 47 (1969); S.L.Adler, Phys. Rev. **177**, 2426 (1969); R.Jackiw and K.Johnson, Phys. Rev. **182**, 1459 (1969); S.L.Adler and W.Bardeen, Phys. Rev. **182**, 1517 (1969); C.R.Hagen, Phys. Rev. **177**, 2622 (1969); B. Zumino, *Proceedings of the Topical conference on Weak Interactions*, CERN, Geneva 1969), p.361.
6. M.Baker and K.Johnson, Phys. Rev. **D3**, 2516 (1971).
7. A.Salam, Phys. Rev. **130**, 1287 (1963); R.Delburgo and P. West, Phys. Lett. **B72**, 3413; *ibid* J.Phys. **A** **10**, 1049 (1977). See also D.W.Atkinson and H.A.Slim, Nuovo Cimento **A50**, 555 (1979).
8. J.Ball and F.Zachariasen, Phys. Lett.**B** **106**, 133 (1981); J.Ball and T.Chieu, Phys. Rev. **D** **22**, 2542 (1980); J. Cornwall, *Phys. Rev. D* **26**(1983), 1453.

9. C.Bender et al, preprint arXiv:hep-th/0501180, January 2005; C.Bender and S. Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998); C. Bender et al, *Phys. Rev. Lett.***93**, 251601 (2004); C. Bender et al, preprint arXiv:hep-th/0411064, Nov. 2004; other references cited therein. See also M.E.Peskin and D.V.Schroeder, *An Introduction to Field theory*, p.425, Addison-Wesley Publishing (1995) New York.
10. S.Weinberg, *The Quantum theory of Fields*, Vol.II, p.154, Cambridge University Press (1996) New York.
11. P. Argyres, M. Plesser, N. Seiberg and E. Witten, *Nucl. Phys.* **B 461**, (1996) 71 .
12. J.Bernstein, *Elementary Particles and their Currents*, W.H.Freeman and Co., 1968.
13. R.Acharya and P. Narayana Swamy, *Nuovo Cimento* **A103**, 1131 (1990).
14. S. Weinberg, *Phys.Rev.* **118**, 838 (1960).
15. See *e.g.*, S.L.Adler and W.Bardeen, *Phys.Rev.* **D4**, 3045 (1971).
16. C. Callan, *Summer School of Theoretical Physics, Les Houches 1971* editors C.Dewitt and C.Itzykson. Gordon Breach publishers (1973) New York; see also S. Weinberg, *Phys.Rev.* **D8**, 3497 (1973).
17. S.Adler, J.C.Collins and A.Duncan, *Phys. Rev.* **D15**, 1712 (1977).
18. T.Eguchi, *Phys.Rev.* **D17**, 611 (1978).
19. It is more expedient to express the solution in terms of the functions A and Σ : see *e.g.*, R. Acharya and P. Narayana Swamy, *Nuovo Cimento* **A98**, 773 (1987).
20. A.Bashir and R.Delbourgo, preprint arXiv:hep-ph/0405018, May 2004.
21. J.M.Cornwall, *Phys.Rev.* **D 26**, 1453 (1983); J.E.King, *Phys.Rev.***D 27**, 1821 (1983).
22. Some of these results are contained in R.Arnouitt and S.Deser, *Phys. Rev.* **138B** , 712 (1965).
23. J. Zinn-Justin, *Quantum Field theory and critical phenomena*, second edition, Clarendon Press, (1993) Oxford.
24. T.Banks and A.Zaks, *Nucl.Phys* **B 196**,189 (1982). It has been estimated in QCD that $N_F^{\min} \approx 12$; see T.Appelquist et al, *Phys.Rev.* **D 58** , 105017 (1998).